Η ΣΥΝΑΡΤΗΣΗ «1-1»

$\text{Η } \textit{συνάρτηση } f : D_f \to \mathbb{R}\left(D_f : \text{Το } \textit{πεδίο ορισμού της } f\right) \textit{ είναι } "1-1" \textit{ όταν}:$ $\textit{Για κάθε } x_1 \neq x_2 \in D_f \textit{ με } x_1 \neq x_2 \textit{ έχω } f(x_1) \neq f(x_2)$
$\text{Η } \textit{συνάρτηση } f : D_f \to \mathbb{R}\left(D_f : \text{Το } \textit{πεδίο ορισμού της } f\right) \textit{ είναι } "1-1" \textit{ όταν}:$ $\textit{Για κάθε } x_1, x_2 \in D_f \textit{ με } f\left(x_1\right) = f\left(x_2\right) \Rightarrow x_1 = x_2$
$\text{Κ}\textit{άθε γνησίως μονότονη συνάρτηση είναι } "1-1"$

ΠΑΡΑΔΕΙΓΜΑΤΑ

1.

$\text{Ν}\textit{α αποδείξετε οτι η συνάρτηση } f\left(x\right) = 2x^3 + 3x + 5 \textit{ είναι } "1-1"$

$D_f = \mathbb{R}$

$\text{Α}\nu \; x_1, x_2 \in \mathbb{R} \textit{ με } x_1 < x_2$

$$x_1 < x_2 \Rightarrow \left\{\begin{matrix} x_1^3 < x_2^3 \\ x_1 < x_2 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} 2x_1^3 < 2x_2^3 \\ 3x_1 < 3x_2 \end{matrix}\right\} \xRightarrow{(+)} 2x_1^3 + 3x_1 < 2x_2^3 + 3x_2 \Rightarrow$$

$$2x_1^3 + 3x_1 + 5 < 2x_2^3 + 3x_2 + 5 \Rightarrow f\left(x_1\right) < f\left(x_2\right)$$

$\Sigma\textit{υνεπώς η } f \textit{ είναι γνησίως αύξουσα. Άρα είναι } "1-1"$

2.

$\text{Ν}\textit{α αποδείξετε οτι η συνάρτηση } f : \left(3, +\infty\right) \to \mathbb{R} \textit{ με τύπο}$ $f\left(x\right) = x^2 - 6x + 7 \textit{ είναι } "1-1"$

$\text{Α}\nu \; x_1, x_2 \in \left(3, +\infty\right) \textit{ με } f\left(x_1\right) = f\left(x_2\right)$

$$\left\{\begin{matrix} f\left(x_1\right) = f\left(x_2\right) \\ x_1, x_2 \in \left(3, +\infty\right) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x_1^2 - 6x_1 \cancel{+7} = x_2^2 - 6x_2 \cancel{+7} \\ x_1, x_2 \in \left(3, +\infty\right) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x_1^2 - x_2^2 - 6x_1 + 6x_2 = 0 \\ x_1, x_2 \in \left(3, +\infty\right) \end{matrix}\right\}$$

$$\Rightarrow \left\{\begin{matrix} \left(x_1 - x_2\right)\left(x_1 + x_2\right) - 6\left(x_1 - x_2\right) = 0 \\ x_1, x_2 \in \left(3, +\infty\right) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \left(x_1 - x_2\right)\left(x_1 + x_2 - 6\right) = 0 \\ x_1, x_2 \in \left(3, +\infty\right) \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{pmatrix} x_1 - x_2 = 0 \\ x_1, x_2 \in \left(3, +\infty\right) \end{pmatrix} \acute{\eta} \begin{pmatrix} x_1 + x_2 - 6 = 0 \\ x_1, x_2 \in \left(3, +\infty\right) \end{pmatrix} = 0\right\} \Rightarrow$$

$$\left\{\begin{pmatrix} x_1 = x_2 \\ x_1, x_2 \in (3,+\infty) \end{pmatrix} \acute{\eta} \begin{pmatrix} x_1 + x_2 - 6 = 0 \\ x_1, x_2 \in (3,+\infty) \end{pmatrix} = 0 \right\} (1)$$

$$\acute{E}\chi\omega\ x_1, x_2 \in (3,+\infty):$$

$$x_1, x_2 \in (3,+\infty) \Rightarrow \left\{\begin{matrix} x_1 > 3 \\ x_2 > 3 \end{matrix}\right\} \overset{(+)}{\Longrightarrow} x_1 + x_2 > 6 \Rightarrow x_1 + x_2 - 6 > 0 \Rightarrow x_1 + x_2 - 6 \neq 0$$

$\text{Επειδή } x_1 + x_2 - 6 \neq 0 \text{ για κάθε } x_1, x_2 \in (3,+\infty) \text{ απο την } (1) \text{ θα έχω } x_1 = x_2$

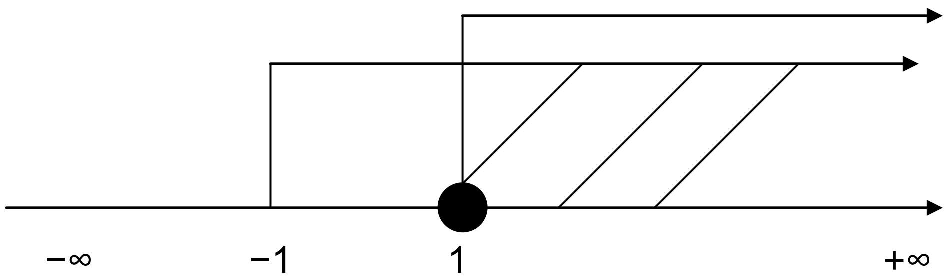
$\text{Συνεπώς η } f \text{ είναι } "1-1"$

3.

$\text{Να αποδείξετε οτι η συνάρτηση } f(x) = \dfrac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} \text{ είναι } "1-1"$

$$D_f = \left\{ x \in \mathbb{R} : x+1 \geq 0, x-1 \geq 0, \sqrt{x+1} + \sqrt{x-1} \neq 0 \right\}$$

$$x \in D_f \Leftrightarrow \left\{\begin{matrix} x+1 \geq 0 \\ x-1 \geq 0 \\ \sqrt{x+1} + \sqrt{x-1} \neq 0 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} x \geq -1 \\ x \geq 1 \\ \sqrt{x+1} + \sqrt{x-1} \neq 0 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} x \geq 1 \\ \sqrt{x+1} + \sqrt{x-1} \neq 0 \end{matrix}\right\}$$



$\text{Θεωρώ την εξίσωση}:$

$$\left\{\begin{matrix} x \geq 1 \\ \sqrt{x+1} + \sqrt{x-1} = 0 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} x \geq 1 \\ \sqrt{x+1} + \sqrt{x-1} = 0 \\ \sqrt{x+1} \geq 0 \\ \sqrt{x-1} \geq 0 \end{matrix}\right\} \overset{\substack{\text{Όταν το άθροισμα δυο μη} \\ \text{αρνητικών αριθμών είναι} \\ \text{μηδέν τότε και οι δυο} \\ \text{αριθμοί είναι μηδέν}}}{\Longleftrightarrow}$$

$$\left\{\begin{matrix} x \geq 1 \\ \sqrt{x+1} = \sqrt{x-1} = 0 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} x \geq 1 \\ x+1 = x-1 = 0 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} x \geq 1 \\ x = -1 \text{ και } x = 1 \\ (\text{Άτοπο}) \end{matrix}\right\}$$

$\Sigma \upsilon \nu \varepsilon \pi \acute{\omega} \varsigma\ \gamma \iota \alpha\ \kappa \acute{\alpha} \theta \varepsilon\ x \in (1,+\infty)\ \theta \alpha\ \iota \sigma \chi \acute{\upsilon} \varepsilon \iota\ \sqrt{x+1}+\sqrt{x-1} \neq 0$

$$x \in D_f \Leftrightarrow \left.\begin{cases} x \geq 1 \\ \sqrt{x+1}+\sqrt{x-1} \neq 0 \end{cases}\right\} \Leftrightarrow x \geq 1 \Leftrightarrow x \in [1,+\infty)$$

$O\pi \acute{o} \tau \varepsilon\ D_f = [1,+\infty)$

$A\nu\ x_1, x_2 \in [1,+\infty)\ f(x_1) = f(x_2):$

$$f(x_1) = f(x_2) \Rightarrow \frac{\sqrt{x_1+1}-\sqrt{x_1-1}}{\sqrt{x_1+1}+\sqrt{x_1-1}} = \frac{\sqrt{x_2+1}-\sqrt{x_2-1}}{\sqrt{x_2+1}+\sqrt{x_2-1}} \Rightarrow$$

$$\left(\sqrt{x_1+1}-\sqrt{x_1-1}\right)\left(\sqrt{x_2+1}+\sqrt{x_2-1}\right) = \left(\sqrt{x_1+1}+\sqrt{x_1-1}\right)\left(\sqrt{x_2+1}-\sqrt{x_2-1}\right) \Rightarrow$$

$$\cancel{\sqrt{x_1+1}\sqrt{x_2+1}} + \sqrt{x_1+1}\sqrt{x_2-1} - \sqrt{x_1-1}\sqrt{x_2+1} - \cancel{\sqrt{x_1-1}\sqrt{x_2-1}} =$$

$$\cancel{\sqrt{x_1+1}\sqrt{x_2+1}} - \sqrt{x_1+1}\sqrt{x_2-1} + \sqrt{x_1-1}\sqrt{x_2+1} - \cancel{\sqrt{x_1-1}\sqrt{x_2-1}} \Rightarrow$$

$$\sqrt{x_1+1}\sqrt{x_2-1} - \sqrt{x_1-1}\sqrt{x_2+1} = \sqrt{x_1-1}\sqrt{x_2+1} - \sqrt{x_1+1}\sqrt{x_2-1} \Rightarrow$$

$$\sqrt{x_1+1}\sqrt{x_2-1} + \sqrt{x_1+1}\sqrt{x_2-1} = \sqrt{x_1-1}\sqrt{x_2+1} + \sqrt{x_1-1}\sqrt{x_2+1} \Rightarrow$$

$$\cancel{2}\sqrt{x_1+1}\sqrt{x_2-1} = \cancel{2}\sqrt{x_1-1}\sqrt{x_2+1} \Rightarrow \sqrt{(x_1+1)(x_2-1)} = \sqrt{(x_1-1)(x_2+1)} \Rightarrow$$

$$\left(\sqrt{(x_1+1)(x_2-1)}\right)^2 = \left(\sqrt{(x_1-1)(x_2+1)}\right)^2 \Rightarrow (x_1+1)(x_2-1) = (x_1-1)(x_2+1) \Rightarrow$$

$$\cancel{x_1x_2} - x_1 + x_2 \cancel{-1} = \cancel{x_1x_2} + x_1 - x_2 \cancel{-1} \Rightarrow x_2 + x_2 = x_1 + x_1 \Rightarrow 2x_2 = 2x_1 \Rightarrow$$

$x_1 = x_2$

$\Sigma \upsilon \nu \varepsilon \pi \acute{\omega} \varsigma\ \eta\ \sigma \upsilon \nu \acute{\alpha} \rho \tau \eta \sigma \eta\ f\ \varepsilon \acute{\iota} \nu \alpha \iota$ "1−1"

4.

$N\alpha\ \alpha\pi o\delta\varepsilon\acute{\iota}\xi\varepsilon\tau\varepsilon\ o\tau\iota\ \eta\ \sigma\upsilon\nu\acute{\alpha}\rho\tau\eta\sigma\eta\ f(x) = \begin{cases} 5x, x \leq 0 \\ x^2+3x, x > 0 \end{cases}\ \varepsilon\acute{\iota}\nu\alpha\iota$ "1−1"

$A\nu\ x_1, x_2 \in \mathbb{R}\ \mu\varepsilon\ x_1 < x_2. \Delta\iota\alpha\kappa\rho\acute{\iota}\nu\omega\ \tau\iota\varsigma\ \pi\varepsilon\rho\iota\pi\tau\omega\sigma\varepsilon\iota\varsigma:$

$$\left\{\begin{array}{l} (\mathrm{I})\, x_1, x_2 \in (-\infty, 0]\ \mu\varepsilon\ x_1 < x_2 \\ (\mathrm{II})\, x_1, x_2 \in (0,+\infty)\ \mu\varepsilon\ x_1 < x_2 \\ (\mathrm{III})\, x_1 \in (-\infty,0], x_2 \in (0,+\infty) \end{array}\right\}$$

$\underline{\Pi\rho o\sigma o\chi\acute{\eta}\ \delta\varepsilon\nu\ \mu\pi o\rho\varepsilon\acute{\iota}\ \nu\alpha\ \iota\sigma\chi\acute{\upsilon}\varepsilon\iota\ x_2 \in (-\infty,0], x_1 \in (0,+\infty)\ \gamma\iota\alpha\tau\iota\ x_1 < x_2\,!!!}$

$\Pi \varepsilon \rho \acute{\iota} \pi \tau \omega \sigma \eta \left(\mathrm{I} \right):$

$$\left. \begin{cases} x_1 < x_2 \\ x_1, x_2 \in \left(-\infty, 0 \right] \end{cases} \right\} \Rightarrow \left. \begin{cases} 5x_1 < 5x_2 \\ x_1, x_2 \in \left(-\infty, 0 \right] \end{cases} \right\} \Rightarrow f\left(x_1 \right) < f\left(x_2 \right)$$

$\Pi \varepsilon \rho \acute{\iota} \pi \tau \omega \sigma \eta \left(\mathrm{II} \right):$

$$\left. \begin{cases} x_1 < x_2 \\ x_1, x_2 \in \left(0, +\infty \right) \end{cases} \right\} \Rightarrow \left. \begin{cases} {x_1}^2 < {x_2}^2 \\ 3x_1 < 3x_2 \\ x_1, x_2 \in \left(-\infty, 0 \right] \end{cases} \right\} \Rightarrow \left. \begin{cases} {x_1}^2 + 3x_1 < {x_2}^2 + 3x_2 \\ x_1, x_2 \in \left(-\infty, 0 \right] \end{cases} \right\} \Rightarrow f\left(x_1 \right) < f\left(x_2 \right)$$

$\Pi \varepsilon \rho \acute{\iota} \pi \tau \omega \sigma \eta \left(\mathrm{III} \right):$

$$\left. \begin{cases} x_1 \in \left(-\infty, 0 \right] \\ x_2 \in \left(0, +\infty \right) \end{cases} \right\} \Rightarrow \left. \begin{cases} x_1 \le 0 \\ x_2 > 0 \end{cases} \right\} \Rightarrow \left. \begin{cases} 5x_1 \le 0 \\ {x_2}^2 > 0 \\ 3x_2 > 0 \\ x_1 \in \left(-\infty, 0 \right] \\ x_2 \in \left(0, +\infty \right) \end{cases} \right\} \Rightarrow \left. \begin{cases} 5x_1 \le 0 \\ {x_2}^2 + 3x_2 > 0 \\ x_1 \in \left(-\infty, 0 \right] \\ x_2 \in \left(0, +\infty \right) \end{cases} \right\} \Rightarrow \left. \begin{cases} f\left(x_1 \right) \le 0 \\ f\left(x_2 \right) > 0 \end{cases} \right\} \Rightarrow$$

$f\left(x_1 \right) < f\left(x_1 \right)$

$\Sigma \upsilon \nu \varepsilon \pi \acute{\omega} \varsigma\ \eta\ \sigma \upsilon \nu \acute{\alpha} \rho \tau \eta \sigma \eta\ f\ \varepsilon \acute{\iota} \nu \alpha \iota\ \gamma \nu \eta \sigma \acute{\iota} \omega \varsigma\ \alpha \acute{\upsilon} \xi o \upsilon \sigma \alpha. \text{Ά} \rho \alpha\ \varepsilon \acute{\iota} \nu \alpha \iota\ "1-1"$

5.

$\mathrm{N} \alpha\ \alpha \pi o \delta \varepsilon \acute{\iota} \xi \varepsilon \tau \varepsilon\ o \tau \iota\ \eta\ \sigma \upsilon \nu \acute{\alpha} \rho \tau \eta \sigma \eta\ f\left(x \right) = \begin{cases} -\sqrt{36 - x^2}, x \in \left(0, 6 \right] \\ x^2 + 10x, x \in \left(6, +\infty \right) \end{cases}$

$\varepsilon \acute{\iota} \nu \alpha \iota\ "1-1"$

$D_f = \left(0, 6 \right] \cup \left(6, +\infty \right) = \left(0, +\infty \right)$

$\mathrm{A} \nu\ x_1, x_2 \in \left(0, +\infty \right)\ \mu \varepsilon\ x_1 < x_2 . \Delta \iota \alpha \kappa \rho \acute{\iota} \nu \omega\ \tau \iota \varsigma\ \pi \varepsilon \rho \iota \pi \tau \omega \sigma \varepsilon \iota \varsigma :$

$$\left. \begin{cases} \left(\mathrm{I} \right) x_1, x_2 \in \left(0, 6 \right]\ \mu \varepsilon\ x_1 < x_2 \\ \left(\mathrm{II} \right) x_1, x_2 \in \left(6, +\infty \right)\ \mu \varepsilon\ x_1 < x_2 \\ \left(\mathrm{III} \right) x_1 \in \left(0, 6 \right], x_2 \in \left(6, +\infty \right) \end{cases} \right\}$$

$\underline{\Pi \rho o \sigma o \chi \acute{\eta}\ \delta \varepsilon \nu\ \mu \pi o \rho \varepsilon \acute{\iota}\ \nu \alpha\ \iota \sigma \chi \acute{\upsilon} \varepsilon \iota\ x_2 \in \left(6, +\infty \right), x_1 \in \left(0, 6 \right]\ \gamma \iota \alpha \tau \iota\ x_1 < x_2 !!!}$

$\Pi\varepsilon\rho i\pi\tau\omega\sigma\eta\left(\mathrm{I}\right)$

$$\left.\begin{cases}x_1<x_2\\ x_1,x_2\in(0,6]\end{cases}\right\}\Rightarrow\left.\begin{cases}x_1^{\ 2}<x_2^{\ 2}\\ 0<x_1\le 6\\ 0<x_2\le 6\end{cases}\right\}\Rightarrow\left.\begin{cases}-x_1^{\ 2}>-x_2^{\ 2}\\ x_1^{\ 2}\le 36\\ x_2^{\ 2}\le 36\\ x_1,x_2\in(0,6]\end{cases}\right\}\Rightarrow\left.\begin{cases}36-x_1^{\ 2}>36-x_2^{\ 2}\\ 36-x_1^{\ 2}\ge 0\\ 36-x_2^{\ 2}\ge 0\\ x_1,x_2\in(0,6]\end{cases}\right\}\Rightarrow$$

$$\left.\begin{cases}\sqrt{36-x_1^{\ 2}}>\sqrt{36-x_2^{\ 2}}\\ x_1,x_2\in(0,6]\end{cases}\right\}\Rightarrow\left.\begin{cases}-\sqrt{36-x_1^{\ 2}}<-\sqrt{36-x_2^{\ 2}}\\ x_1,x_2\in(0,6]\end{cases}\right\}\Rightarrow f\left(x_1\right)<f\left(x_2\right)$$

$\Pi\varepsilon\rho i\pi\tau\omega\sigma\eta\left(\mathrm{II}\right)$

$$\left.\begin{cases}x_1<x_2\\ x_1,x_2\in(6,+\infty)\end{cases}\right\}\Rightarrow\left.\begin{cases}x_1^{\ 2}<x_2^{\ 2}\\ 10x_1<10x_2\\ x_1,x_2\in(6,+\infty)\end{cases}\right\}\overset{(+)}{\Longrightarrow}\left.\begin{cases}x_1^{\ 2}+10x_1<x_2^{\ 2}+10x_2\\ x_1,x_2\in(6,+\infty)\end{cases}\right\}\Rightarrow$$

$f\left(x_1\right)<f\left(x_2\right)$

$\Pi\varepsilon\rho i\pi\tau\omega\sigma\eta\left(\mathrm{III}\right)$

$$\left.\begin{cases}x_1\in(0,6]\\ x_2\in(6,+\infty)\end{cases}\right\}\Rightarrow\left.\begin{cases}0<x_1\le 6\\ x_2^{\ 2}>0\\ 10x_2>0\\ x_2\in(6,+\infty)\end{cases}\right\}\Rightarrow\left.\begin{cases}x_1^{\ 2}\le 36\\ x_1\in(0,6]\\ x_2^{\ 2}>0\\ 10x_2>0\\ x_2\in(6,+\infty)\end{cases}\right\}\Rightarrow\left.\begin{cases}36-x_1^{\ 2}\ge 0\\ x_1\in(0,6]\\ x_2^{\ 2}+10x_2>0\\ x_2\in(6,+\infty)\end{cases}\right\}\Rightarrow$$

$$\left.\begin{cases}\sqrt{36-x_1^{\ 2}}\ge 0\\ x_1\in(0,6]\\ f\left(x_2\right)>0\\ x_2\in(6,+\infty)\end{cases}\right\}\Rightarrow\left.\begin{cases}-\sqrt{36-x_1^{\ 2}}\le 0\\ x_1\in(0,6]\\ f\left(x_2\right)>0\\ x_2\in(6,+\infty)\end{cases}\right\}\Rightarrow\left.\begin{cases}f\left(x_1\right)\le 0\\ f\left(x_2\right)>0\end{cases}\right\}\Rightarrow f\left(x_1\right)<f\left(x_2\right)$$

Συνεπώς η συνάρτηση f είναι γνησίως αύξουσα.Άρα είναι "1−1"

6.

Δίνεται η συνάρτηση $f:(0,\infty)\to\mathbb{R}$.*Αν η* f *είναι γνησίως να αποδείξετε οτι η συνάρτηση* $g\left(x\right)=-4f^{3}\left(x\right)+\dfrac{1}{\sqrt{x}},x\in(0,+\infty)$ *είναι* "1−1"

Αν $x_1, x_2 \in (0,+\infty)$ με $x_1 < x_2$

$$\left.\begin{cases} x_1 < x_2 \\ x_1, x_2 \in (0,+\infty) \end{cases}\right\} \overset{f\uparrow(0,+\infty)}{\Longrightarrow} \left.\begin{cases} f(x_1) < f(x_2) \\ \sqrt{x_1} < \sqrt{x_2} \\ x_1, x_2 \in (0,+\infty) \end{cases}\right\} \Rightarrow \left.\begin{cases} f^3(x_1) < f^3(x_2) \\ \dfrac{1}{\sqrt{x_1}} > \dfrac{1}{\sqrt{x_2}} \\ x_1, x_2 \in (0,+\infty) \end{cases}\right\} \Rightarrow$$

$$\left.\begin{cases} -4f^3(x_1) - 4f^3(x_2) \\ \dfrac{1}{\sqrt{x_1}} > \dfrac{1}{\sqrt{x_2}} \\ x_1, x_2 \in (0,+\infty) \end{cases}\right\} \Rightarrow \left.\begin{cases} -4f^3(x_1) + \dfrac{1}{\sqrt{x_1}} > -4f^3(x_2) + \dfrac{1}{\sqrt{x_2}} \\ x_1, x_2 \in (0,+\infty) \end{cases}\right\} \Rightarrow g(x_1) > g(x_2)$$

Οπότε η συνάρτηση g είναι γνησίως φθίνουσα.Άρα είναι "1−1"

ΑΣΚΗΣΕΙΣ

1.

Να αποδείξετε οτι η συνάρτηση $f(x) = (\lambda^4 - \lambda^2 + 3)x^3 + 7x + 2$ είναι "1−1"

Υπόδειξη: $\lambda^4 - \lambda^2 + 3 = \lambda^4 - 2\lambda^2 \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 3 = \left(\lambda^2 - \frac{1}{2}\right)^2 + \frac{11}{4} > 0$

2.

Να αποδείξετε οτι η συνάρτηση $f : (-\infty, 5) \to \mathbb{R}$ με τύπο $f(x) = -x^2 + 10x + 3$ είναι "1−1"

3.

Να αποδείξετε οτι η συνάρτηση $f(x) = \dfrac{\sqrt{x+2} - \sqrt{x-2}}{\sqrt{x+2} + \sqrt{x-2}}$ είναι "1−1"

4.

Να αποδείξετε οτι η συνάρτηση $f(x) = \begin{cases} 7x, x \le 0 \\ x^2 + 9x, x > 0 \end{cases}$ είναι "1−1"

5.

Να αποδείξετε οτι η συνάρτηση $f(x) = \begin{cases} -\sqrt{1-x^2}, x \in (0,1] \\ x^2 + 2x, x \in (1,+\infty) \end{cases}$

είναι "1−1"